Curve-Based Cryptography

Nicolas Thériault

`nicolas.theriault@usach.cl`

Departamento de Matemática y Ciencia de la Computación

Universidad de Santiago de Chile

Discrete Log Problem

Computational Diffie-Hellman Problem: Given g_1, $[a]g_1$, and $[b]g_1$, compute $[ab]g_1$.

For a generic (additive) group G and for well chosen values of a et b, the fastest known method consists in solving the discrete log problem.

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For a generic (additive) group G and for well chosen values of a et b, the fastest known method consists in solving the discrete log problem.

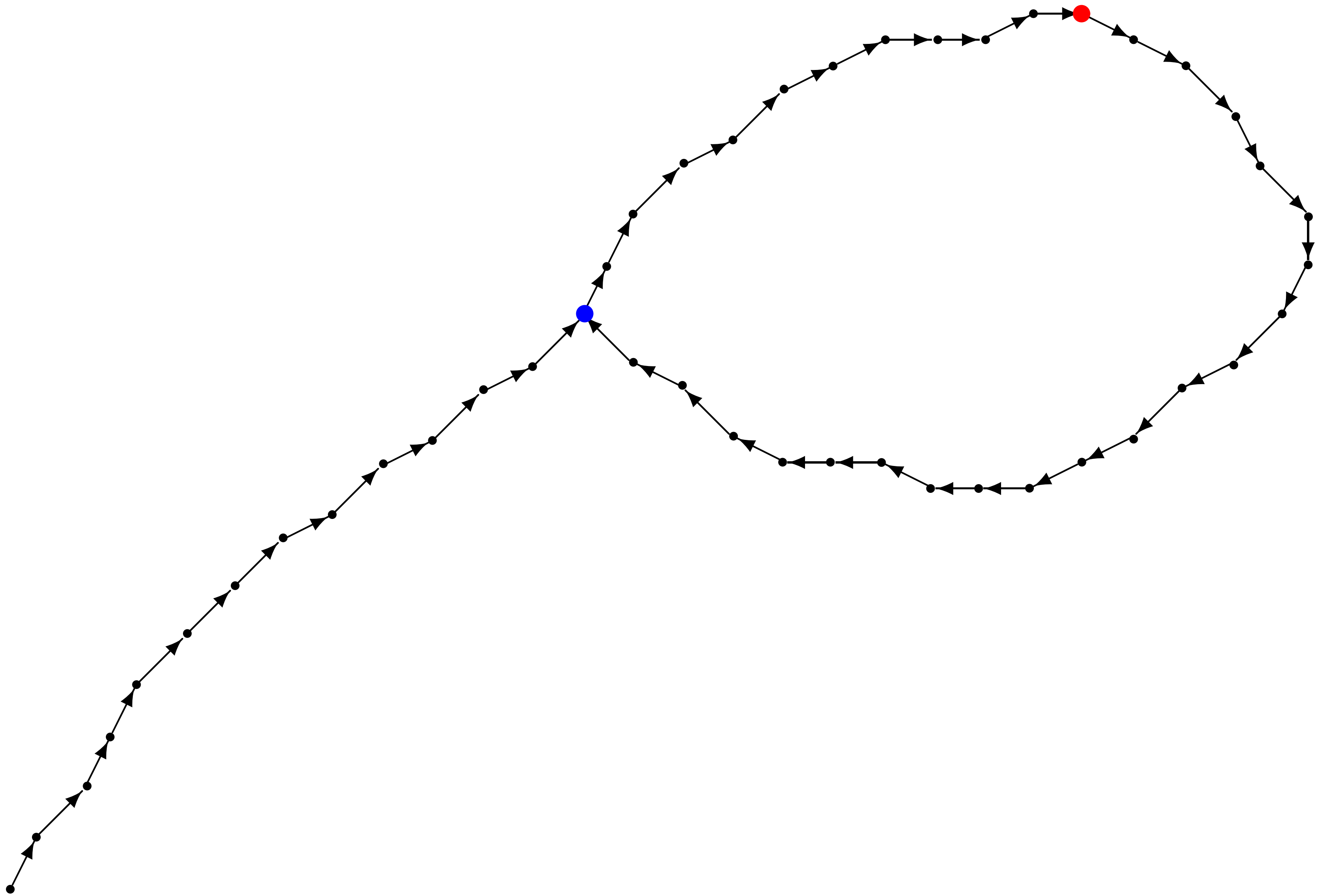
Given two elements g_1 and g_λ of a group G such that $g_\lambda \in \langle g_1 \rangle$, the ***discrete logarithm problem*** for the pair (g_1, g_λ) in G consist in computing the smallest positive integer λ such that $g_\lambda = [\lambda]g_1$.

The security of many public key cryptosystems relies on the difficulty of the discrete log.

Generic Attacks

- Three main types of attack:
  - Shank's Baby Step - Giant Step algorithm;
  - Pollard's ρ method;
  - Pollard's kangaroo method.
- They work for every abelian group.
- They require
$$O\left(\sqrt{\text{group order}}\right)$$
group operations to solve the discrete log.

Example: Pollard's ρ



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For some groups, it's false:

- The additive group $\mathbb{Z}/p\mathbb{Z}$ (we can divide by g_1).
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- Hyperelliptic curves of genus 2 (of prime order).

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For others, it's false, but not too much:

- Hyperelliptic curves of genus 3 and 4.
- Non-hyperelliptic curves of genus 4.

Elliptic Curves

Curve:

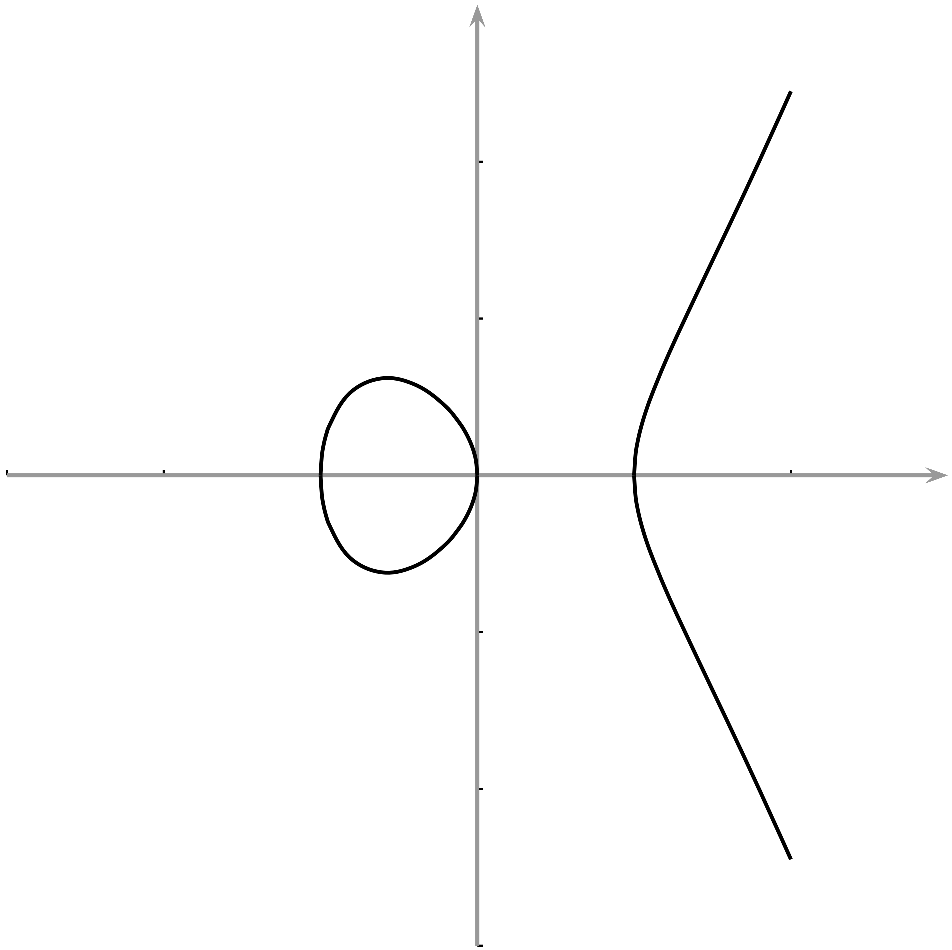
- Has an equation of the form $y^2 = x^3 + ax + b$ (Weierstrass form)
- over a field of q elements, $q = p^k$.
- such that $4a^3 + 27b^2 \neq 0 \bmod p$ (non-singular)

Group:

- The (affine) rational points on the curve of the form (x_i, y_i) where $y_i^2 = x_i^3 + ax_i + b$
- an extra point "at infinity", P_∞, which will be the zero/neutral of the group
- a group operation between pairs of points

Point Addition for $E(\mathbb{R})$

$$y^2 = x^3 - x$$



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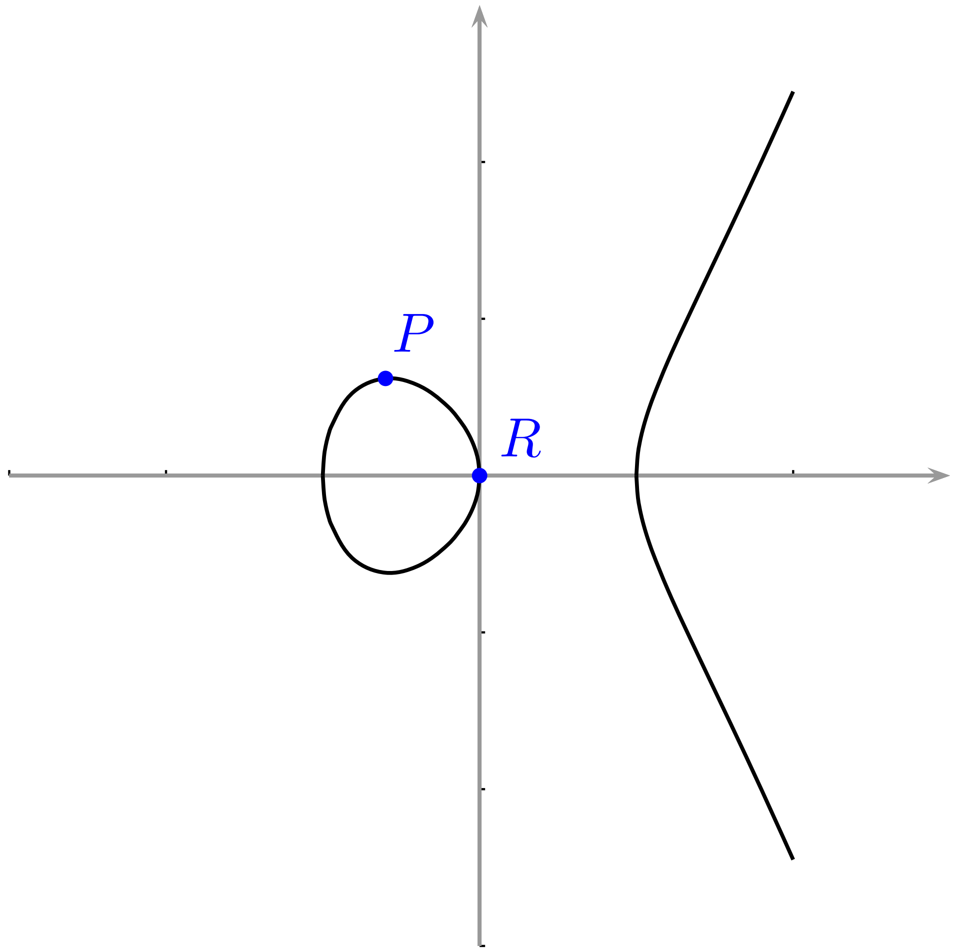

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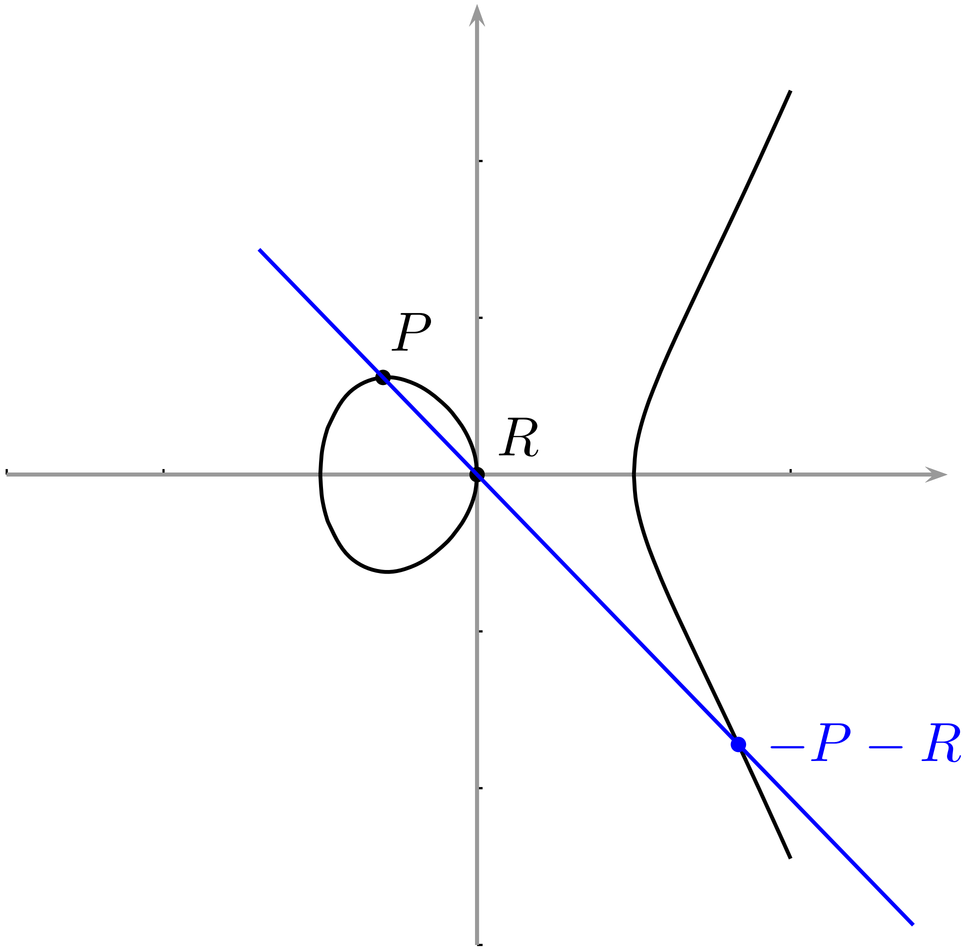

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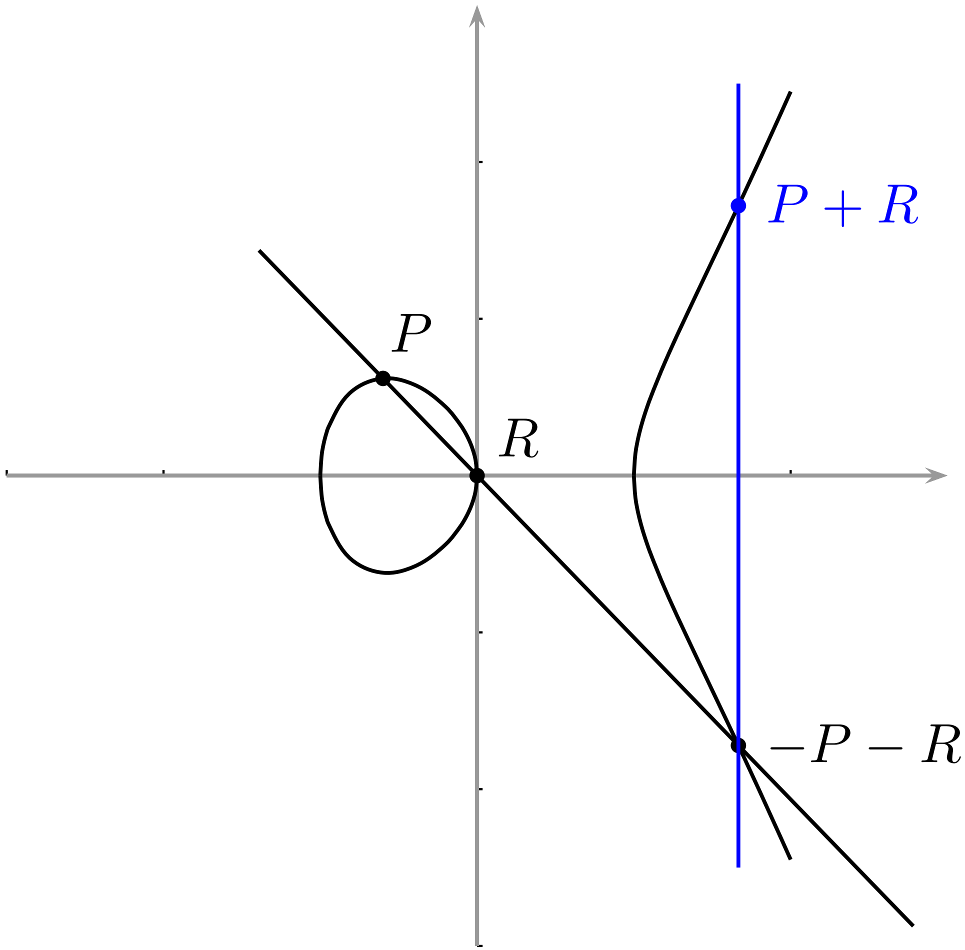


Group operation

Special cases:

- two distinct points on the same vertical add to P_∞
- if the y-coordinate is 0, the double of the point is P_∞
- adding P_∞ to any point returns the same point

General case, the chord-and-tangent method:

- $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$
- $x_3 = \lambda^2 - x_1 - x_2$, $y_3 = -y_1 - \lambda(x_3 - x_1)$
- λ is the slope of the line between the two initial points (of the tangent if both points are the same)
- $\lambda = \frac{y_1 - y_2}{x_1 - x_2}$ (general addition) or $\frac{3x_1^2 + a}{2y_1}$ (doubling)

Varia

- There are other ways to represent elliptic curves, which can give different group operations
- A popular representation is Edwards curves: $x^2 + y^2 = 1 - dx^2y^2$
- Projective coordinates: represent points as triples (or more) of coordinates, to avoid field divisions

maps:

- The complete (extended) group should include all points over the algebraic closure of the field
- Isomorphisms: to change the equation but keep the exact same group
- Isogenies: maps between curves with a finite kernel

Hyperelliptic Curves

A hyperelliptic curve C of genus g is defined by an equation of the form:

$$C : Y^2 + h(X)Y = f(X)$$

with

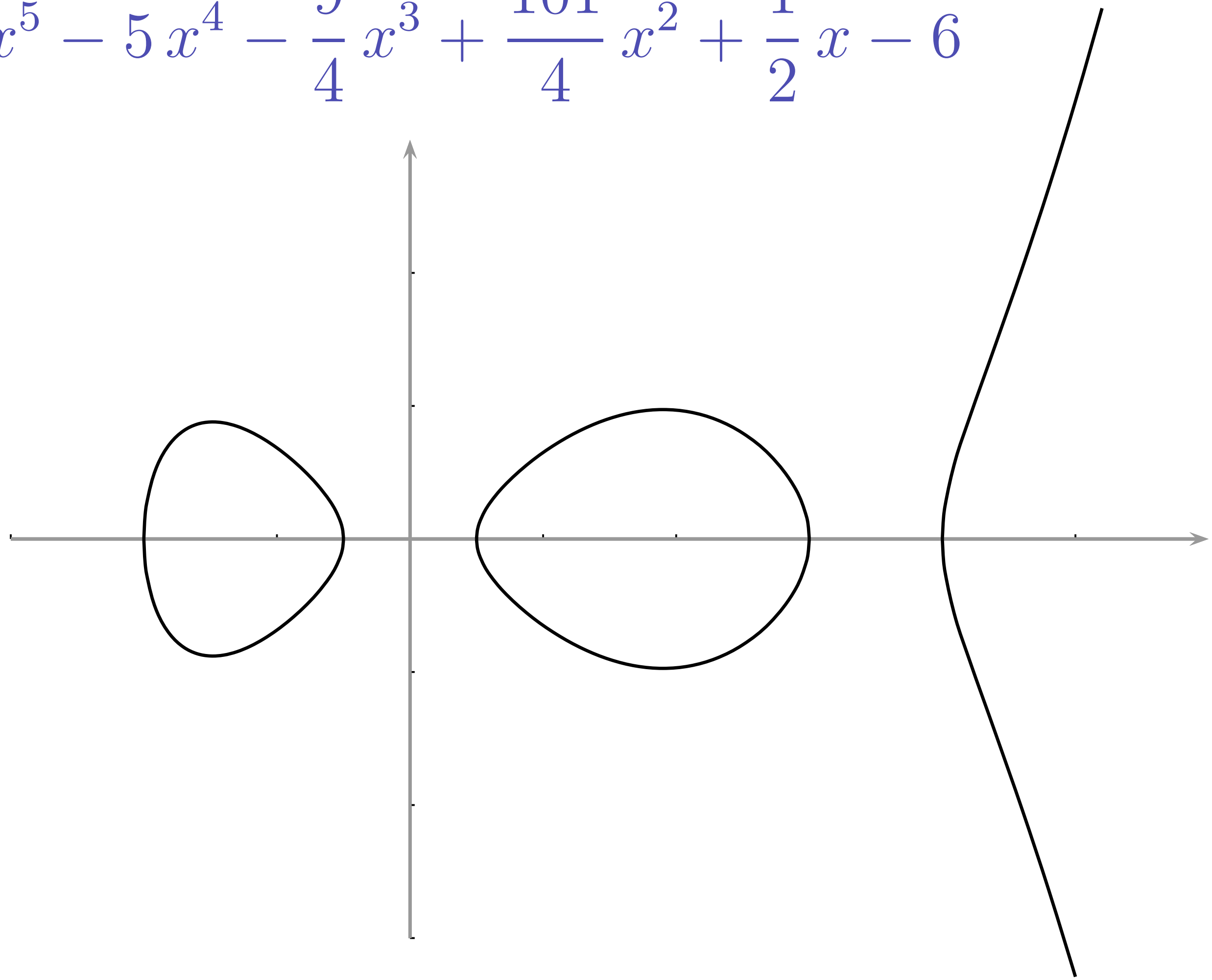
- $\deg(h) \leq g$;
- $\deg(f) = 2g + 1$;
- a tangent to the curve defined at every point.

Elliptic curves are hyperelliptic curves of genus 1.

In genus greater than 1, points do not form a group.

HEC over $\mathbb{R}$, genus 2

$$y^2 = x^5 - 5\,x^4 - \frac{9}{4}\,x^3 + \frac{101}{4}\,x^2 + \frac{1}{2}\,x - 6$$



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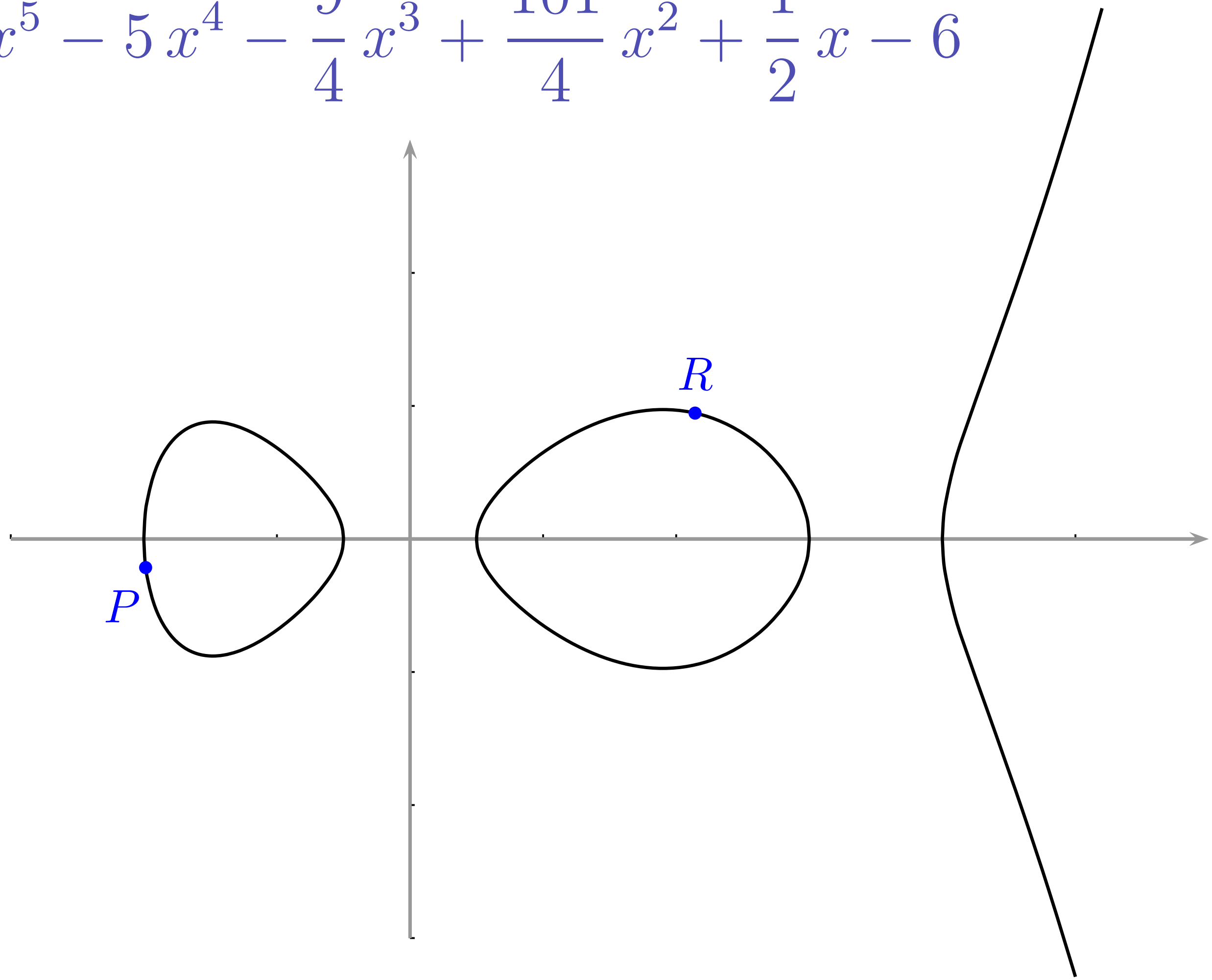

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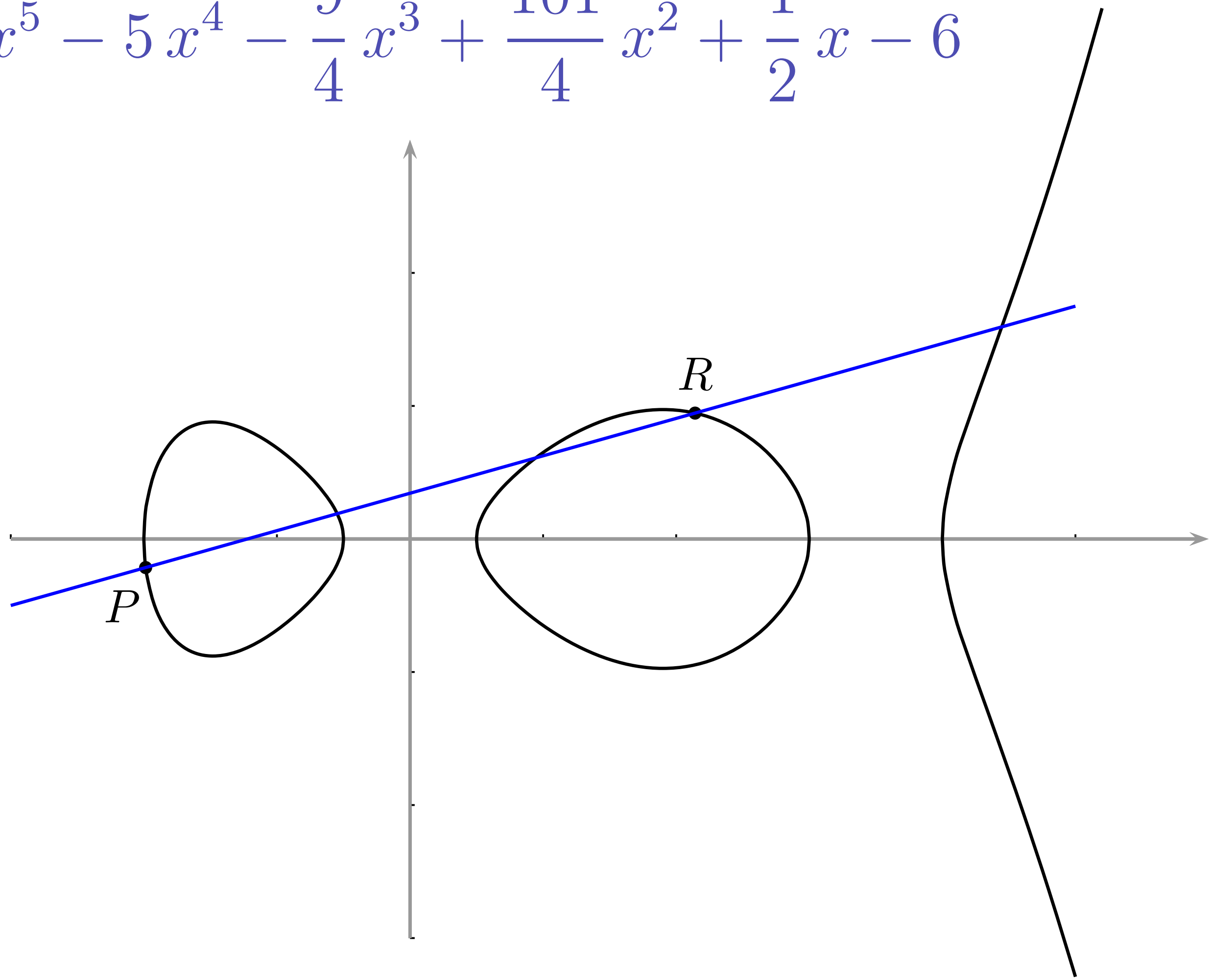

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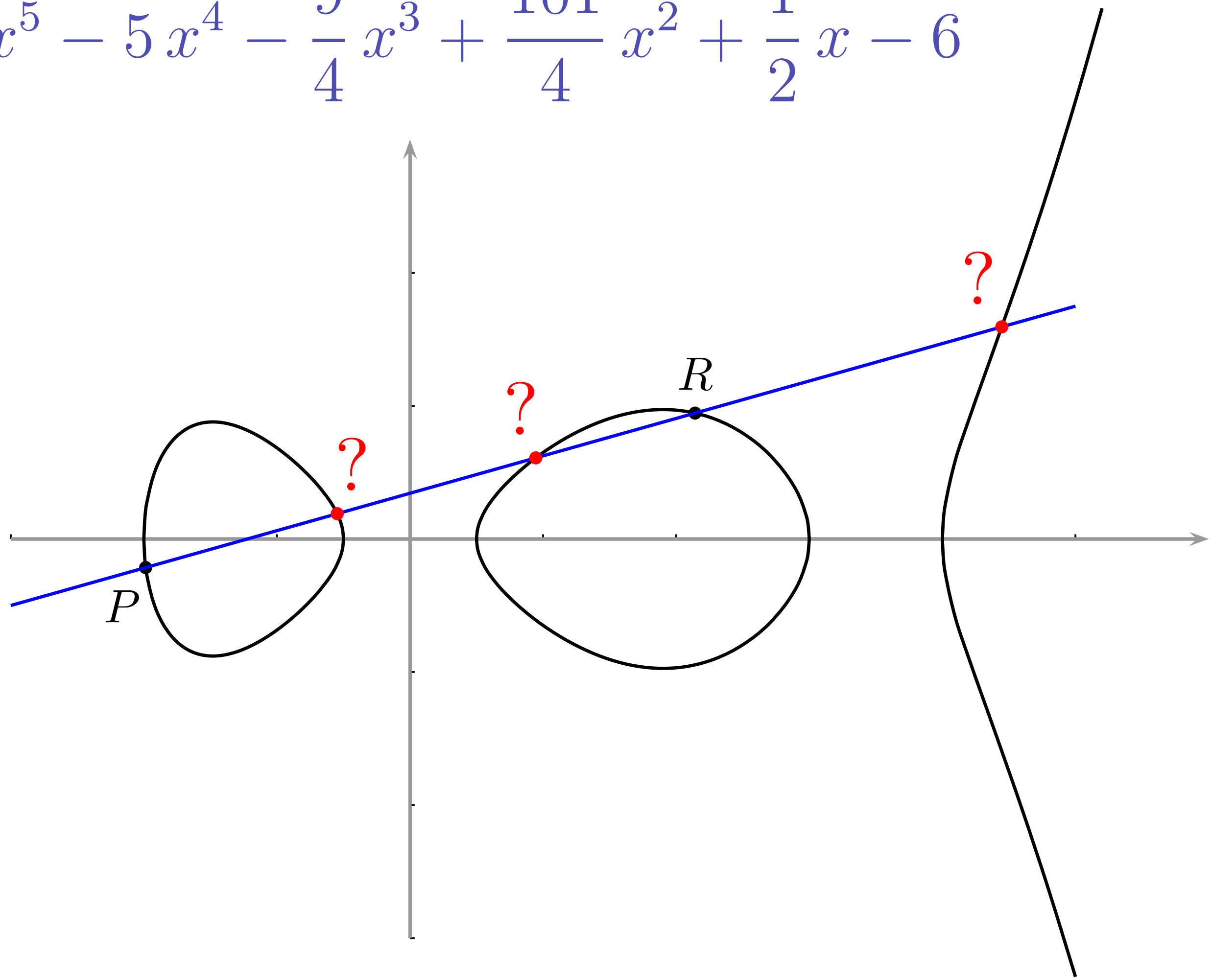


Divisor Class Group

Divisors (sums of points, including ∞) of degree zero ($\sum$ coefficients $= 0$) form an infinite additive group.

A principal divisor is the sum of the points of intersection between the curve and a polynomial in x and y. Principal divisors are a normal subgroup of the divisors of degree zero.

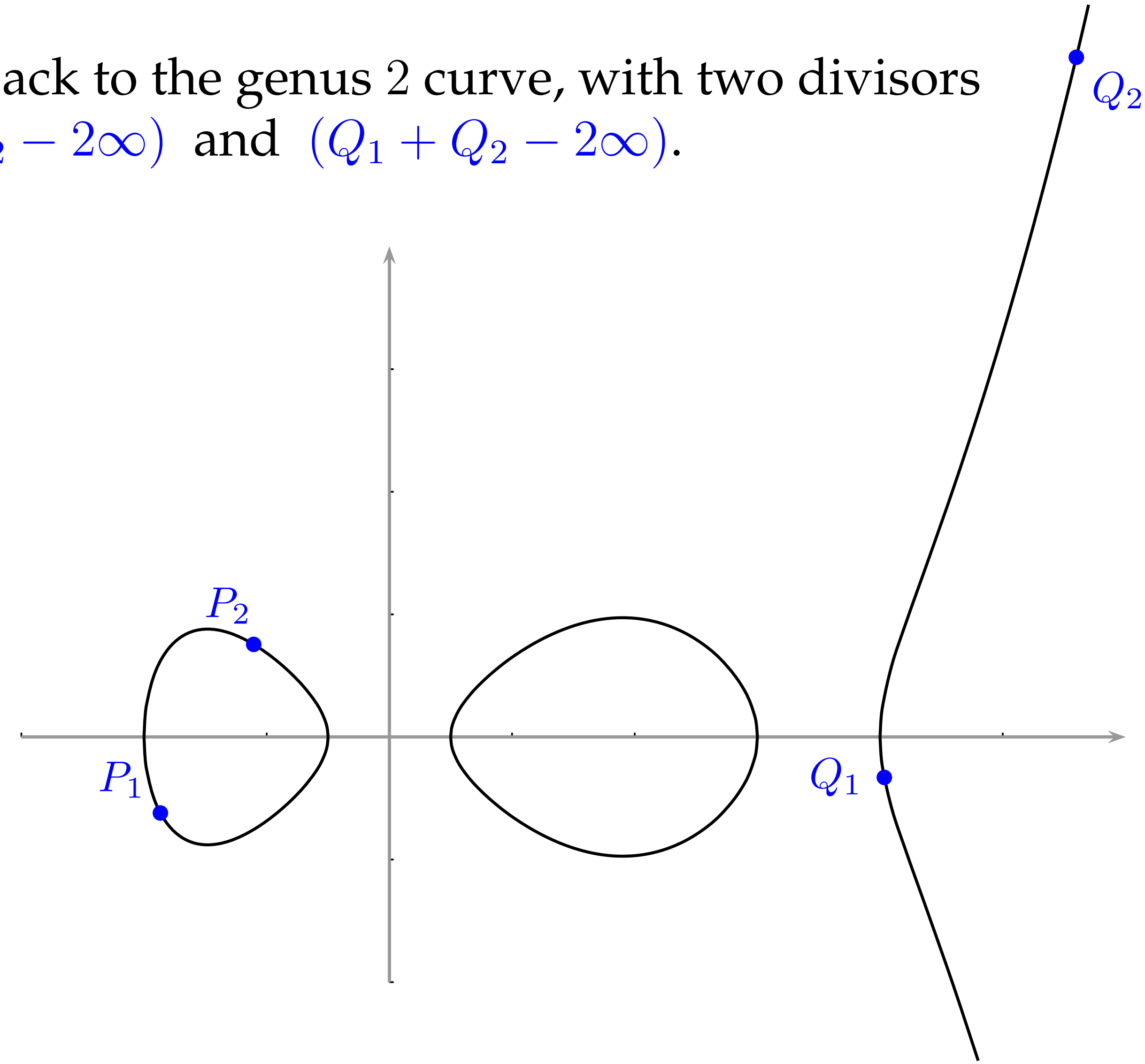
The Jacobian is the group of divisor classes (i.e. divisors of degree zero modulo principal divisors).

A reduced divisor is the sum of at most g points ($-\infty$) and does not contain any pair of points (x, y), $(x, -y - h(x))$.

The element of the Jacobian of C (the divisor classes) are represented by reduced divisors.

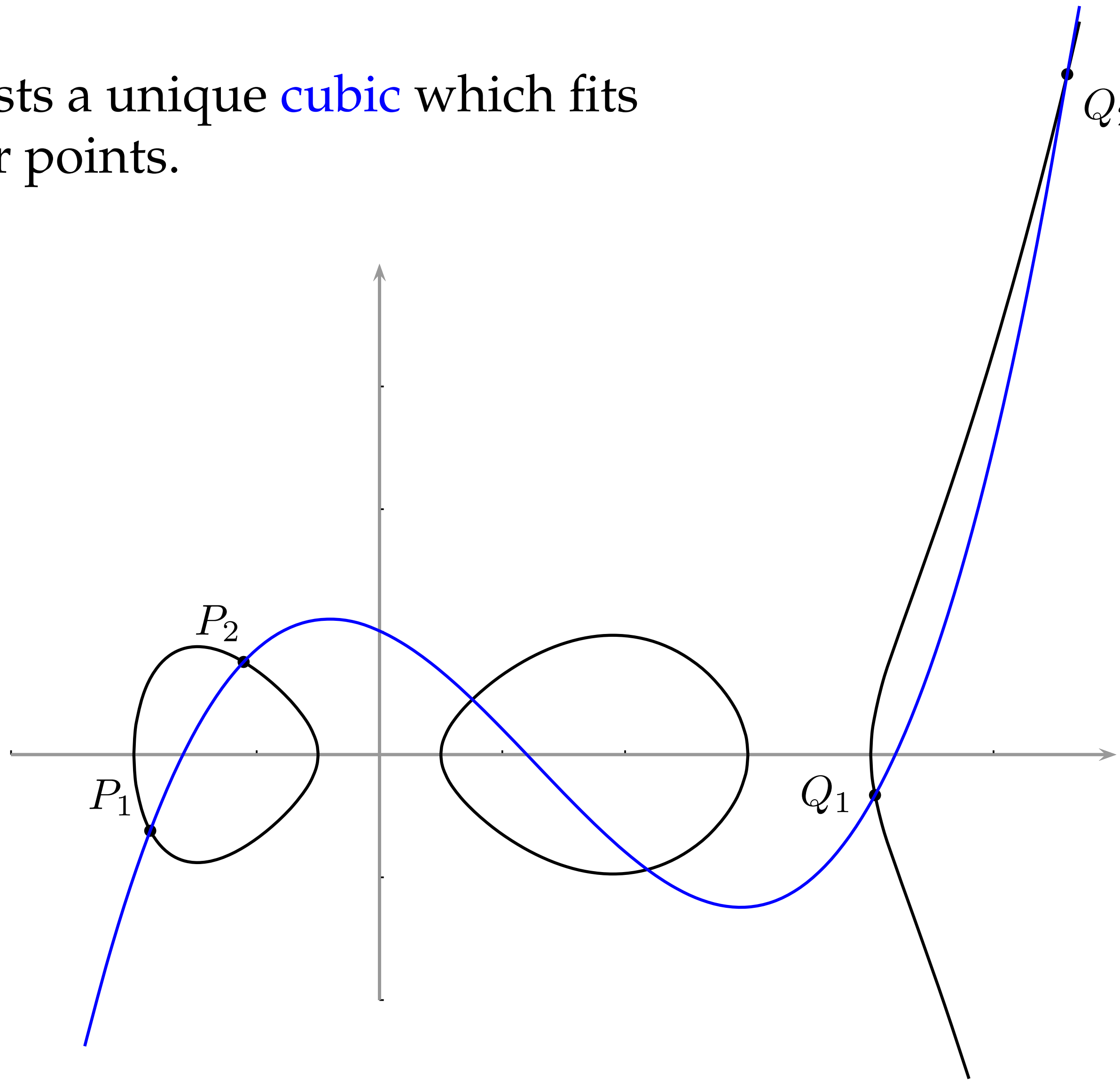
Jacobian Addition

Going back to the genus 2 curve, with two divisors $(P_1 + P_2 - 2\infty)$ and $(Q_1 + Q_2 - 2\infty)$.



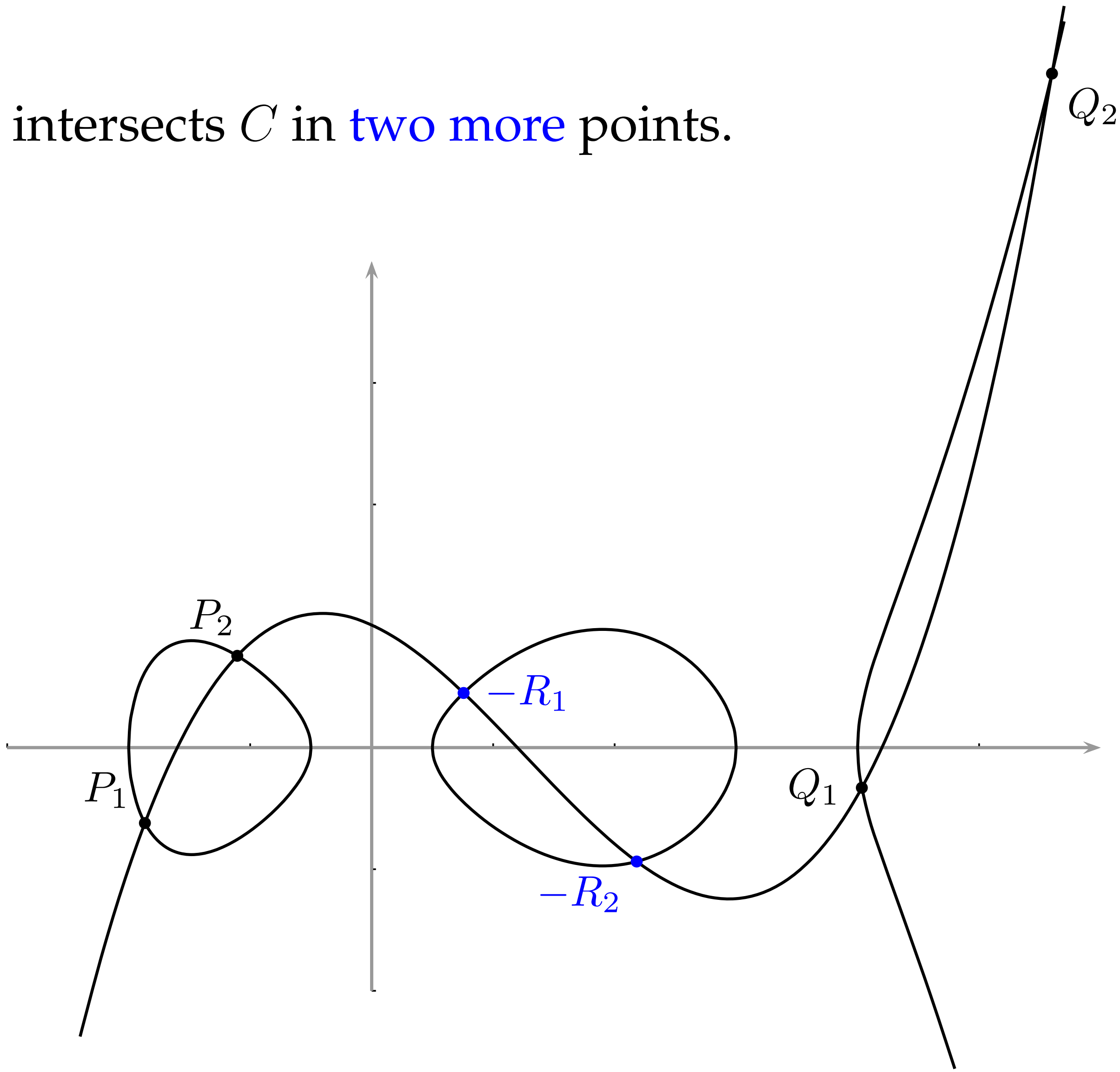
Jacobian Addition

There exists a unique cubic which fits these four points.



Jacobian Addition

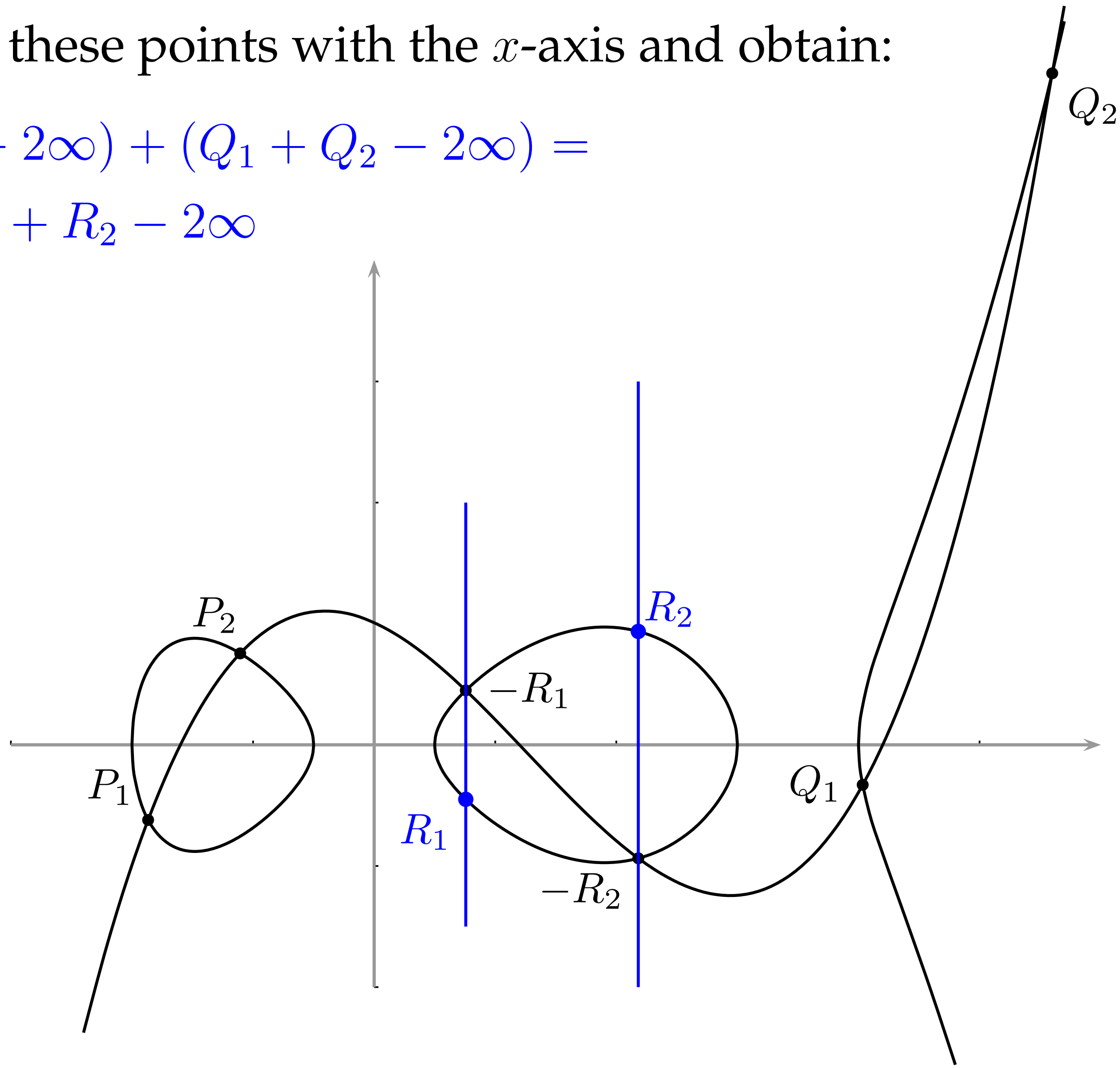
The cubic intersects C in two more points.



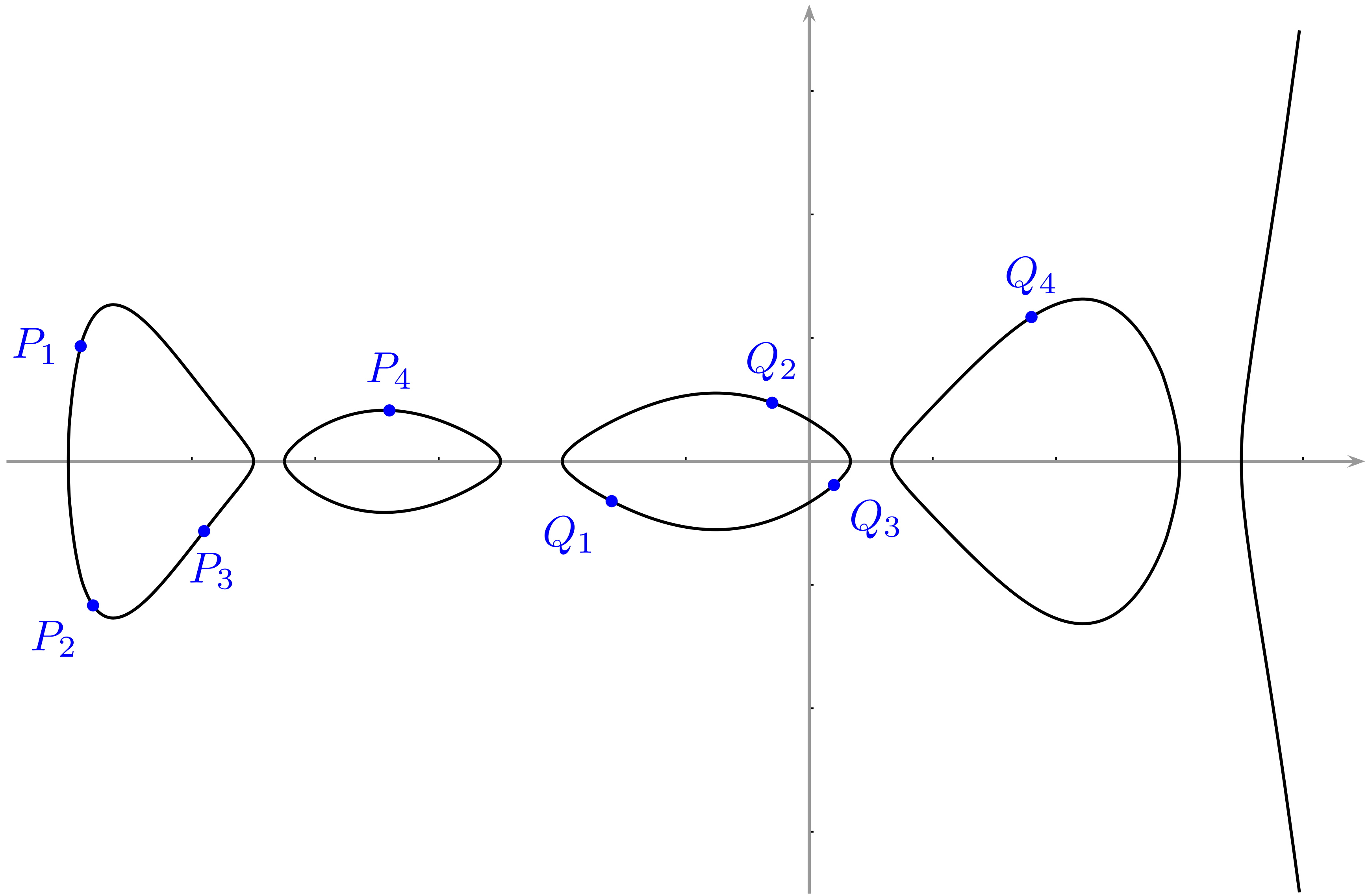
Jacobian Addition

We reflect these points with the x-axis and obtain:

$$(P_1 + P_2 - 2\infty) + (Q_1 + Q_2 - 2\infty) =$$
$$= R_1 + R_2 - 2\infty$$



Curve of Genus 4



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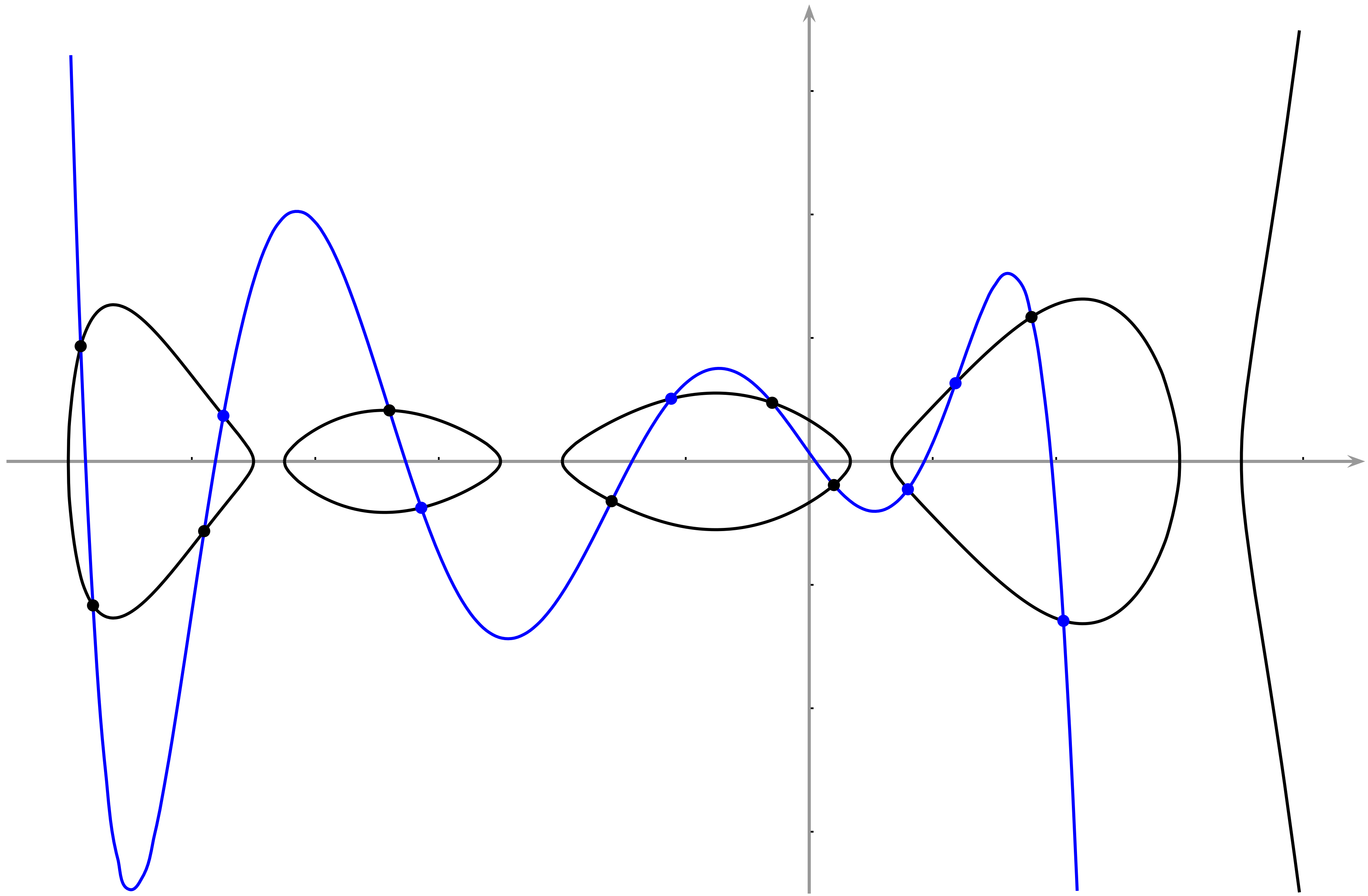

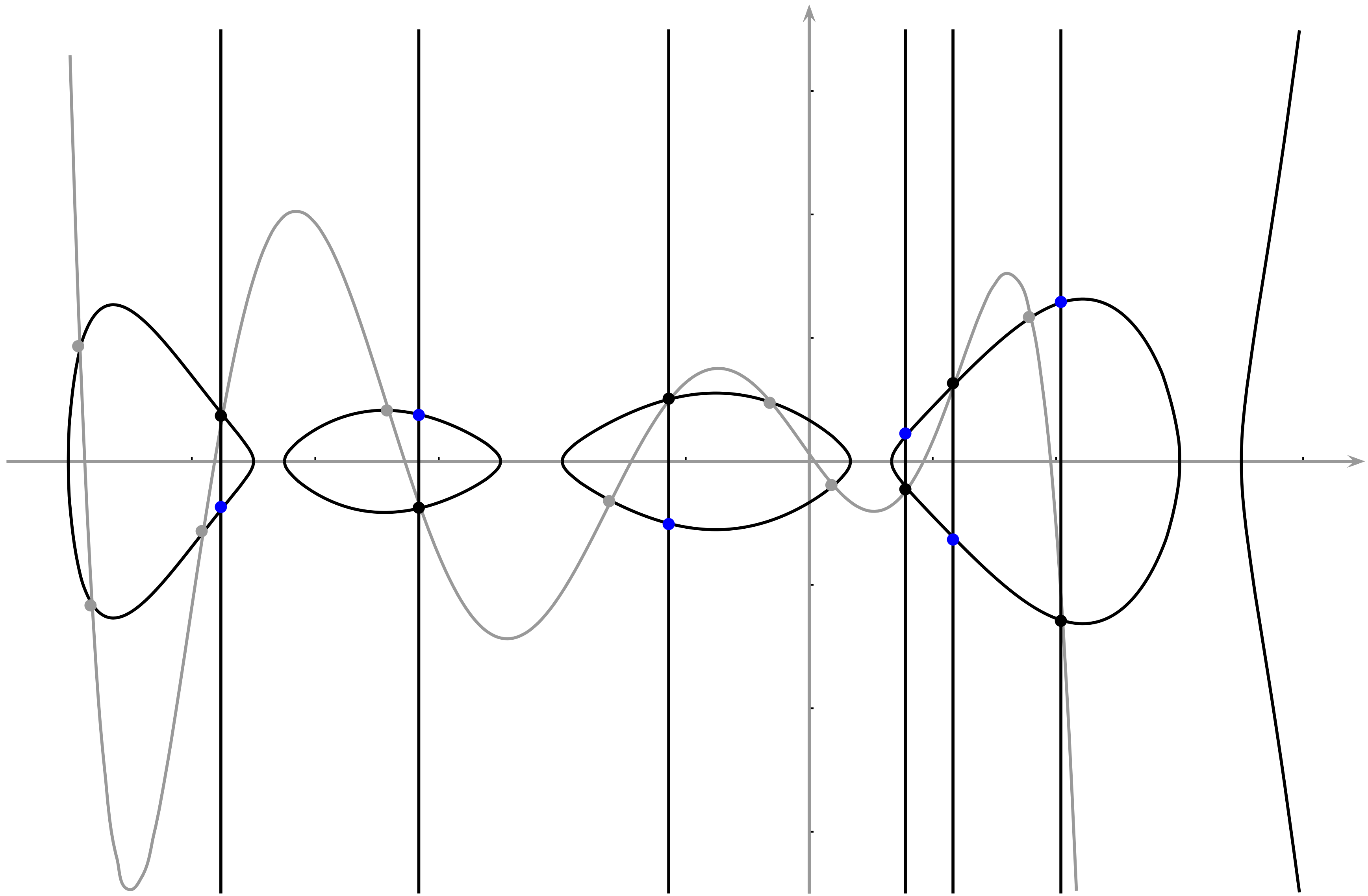

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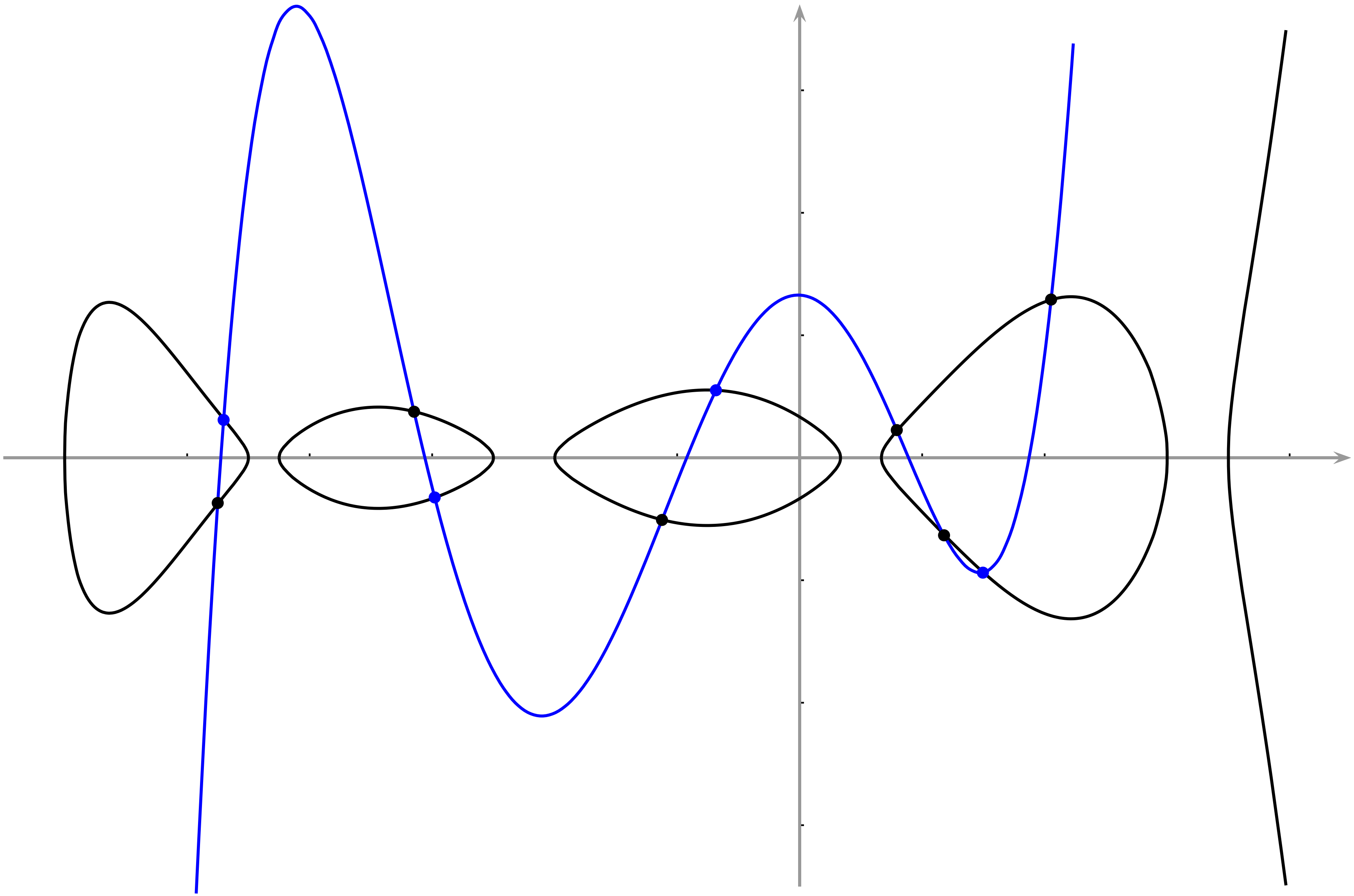

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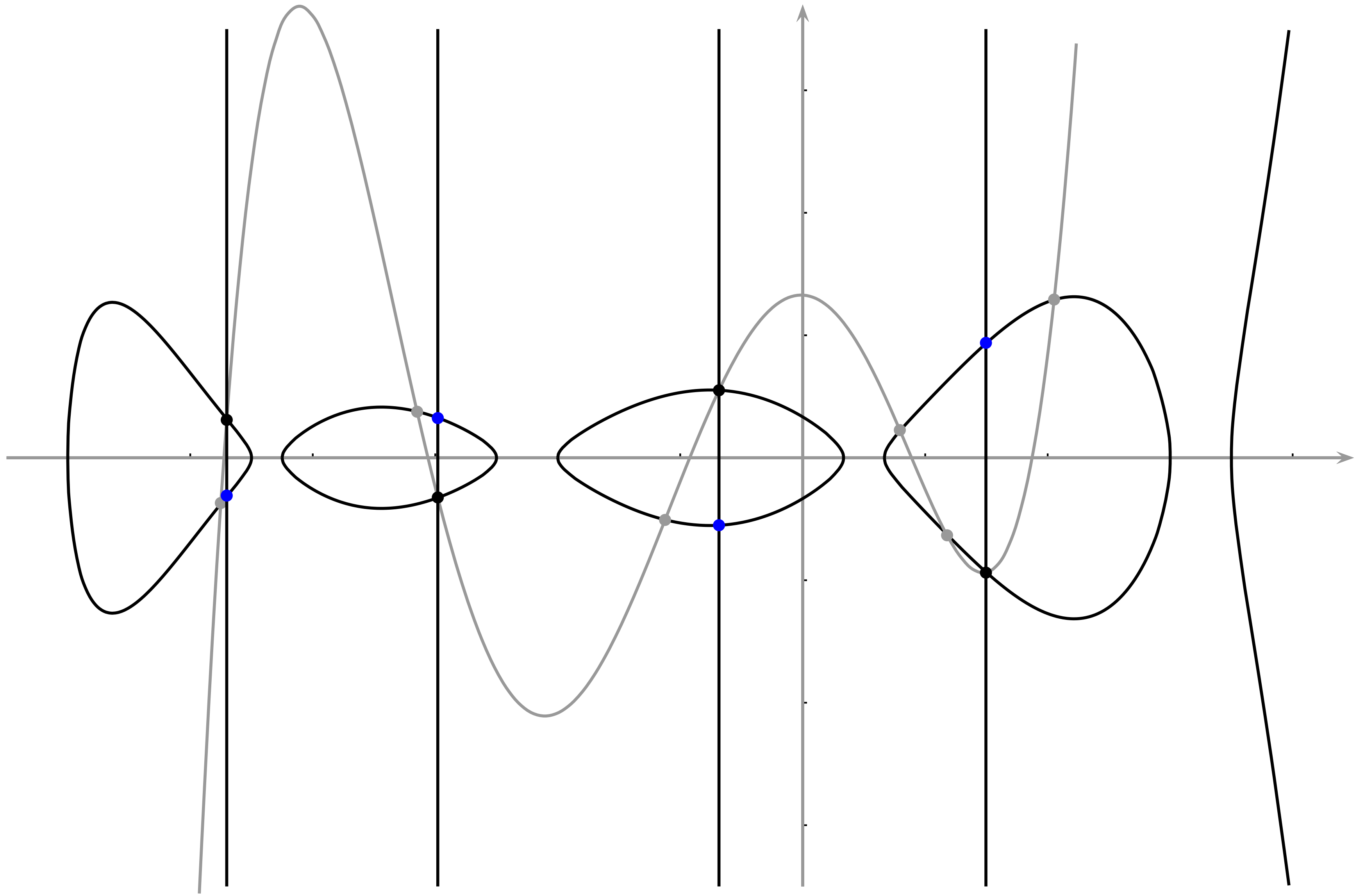

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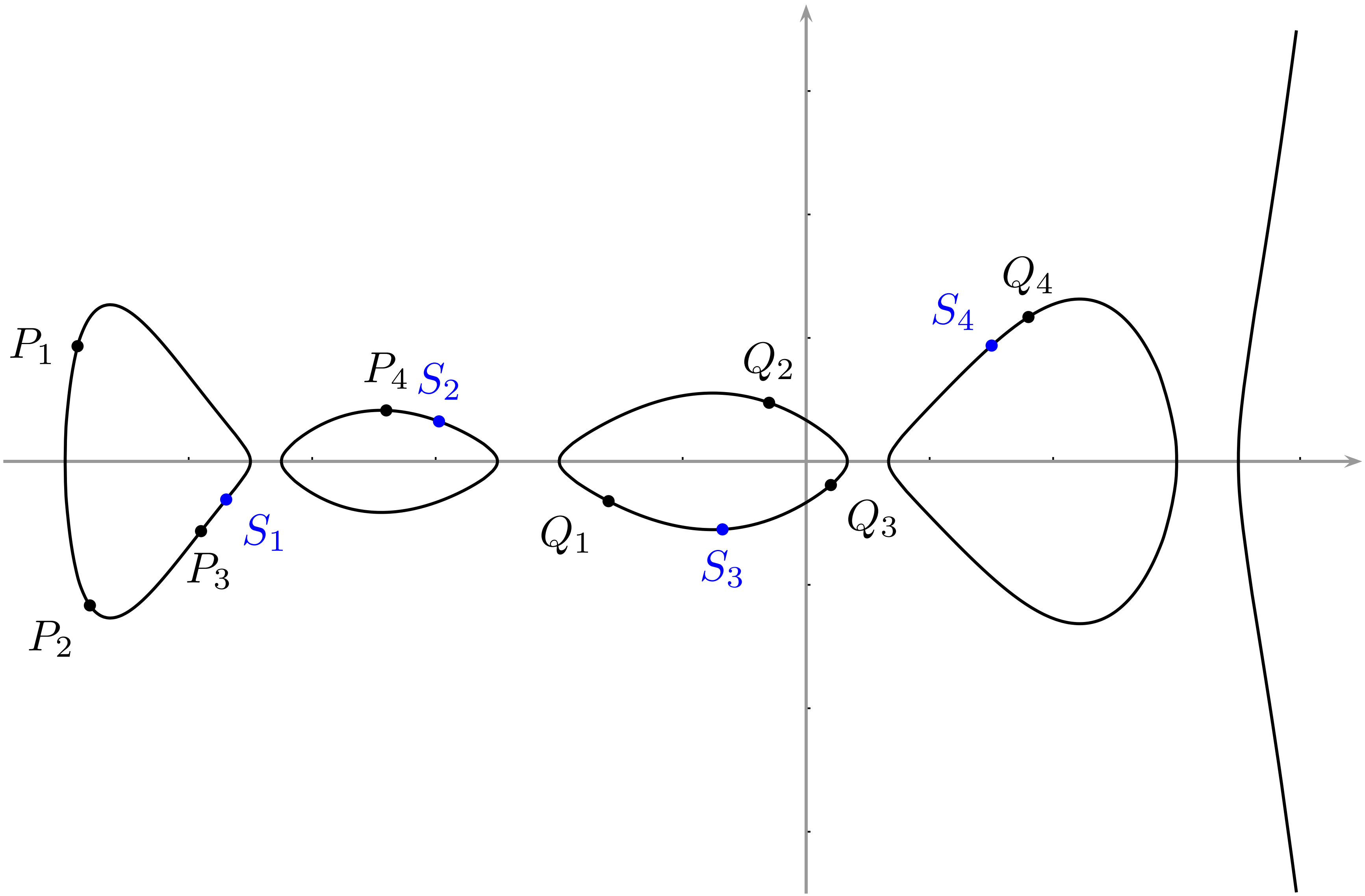
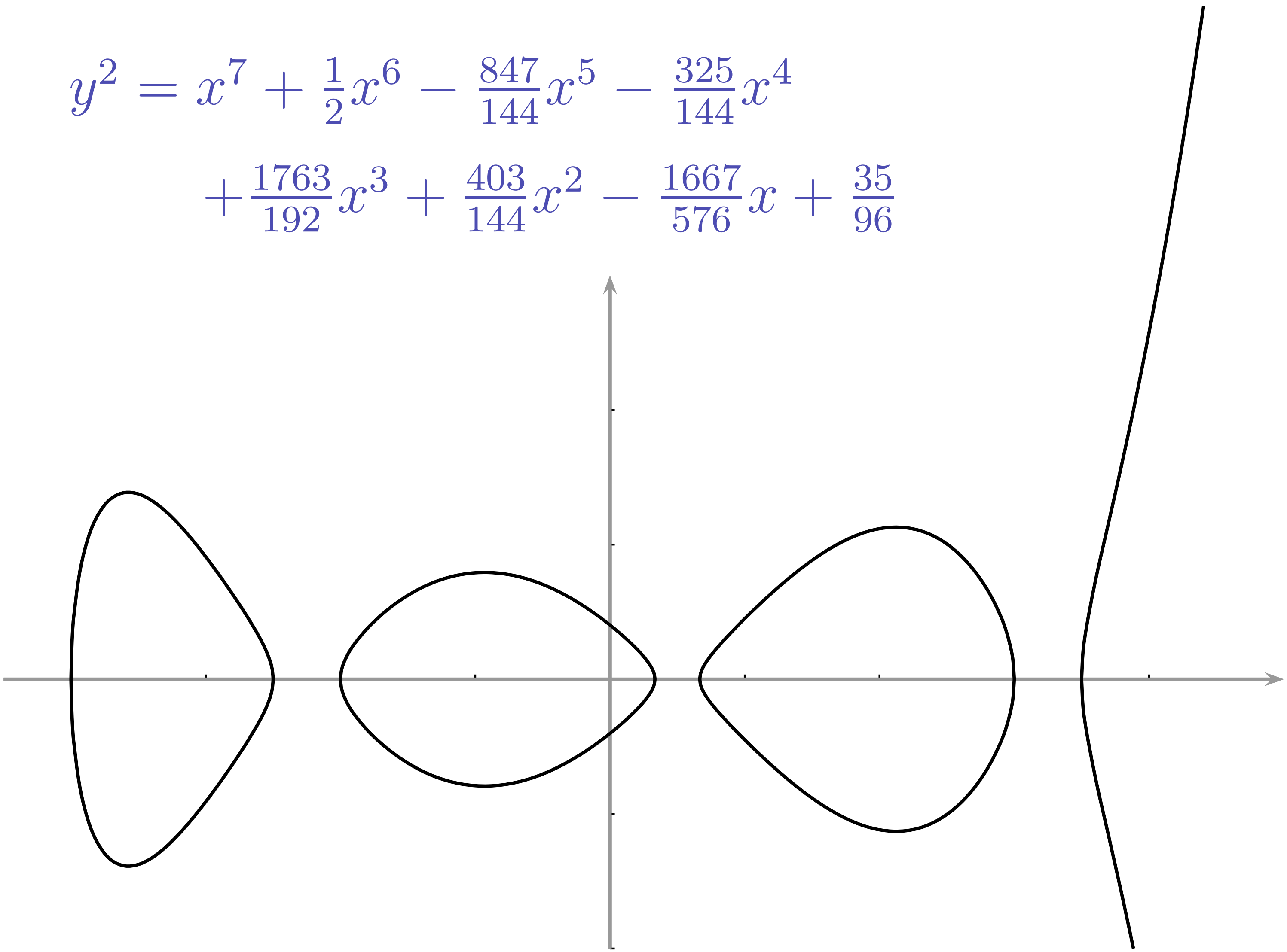
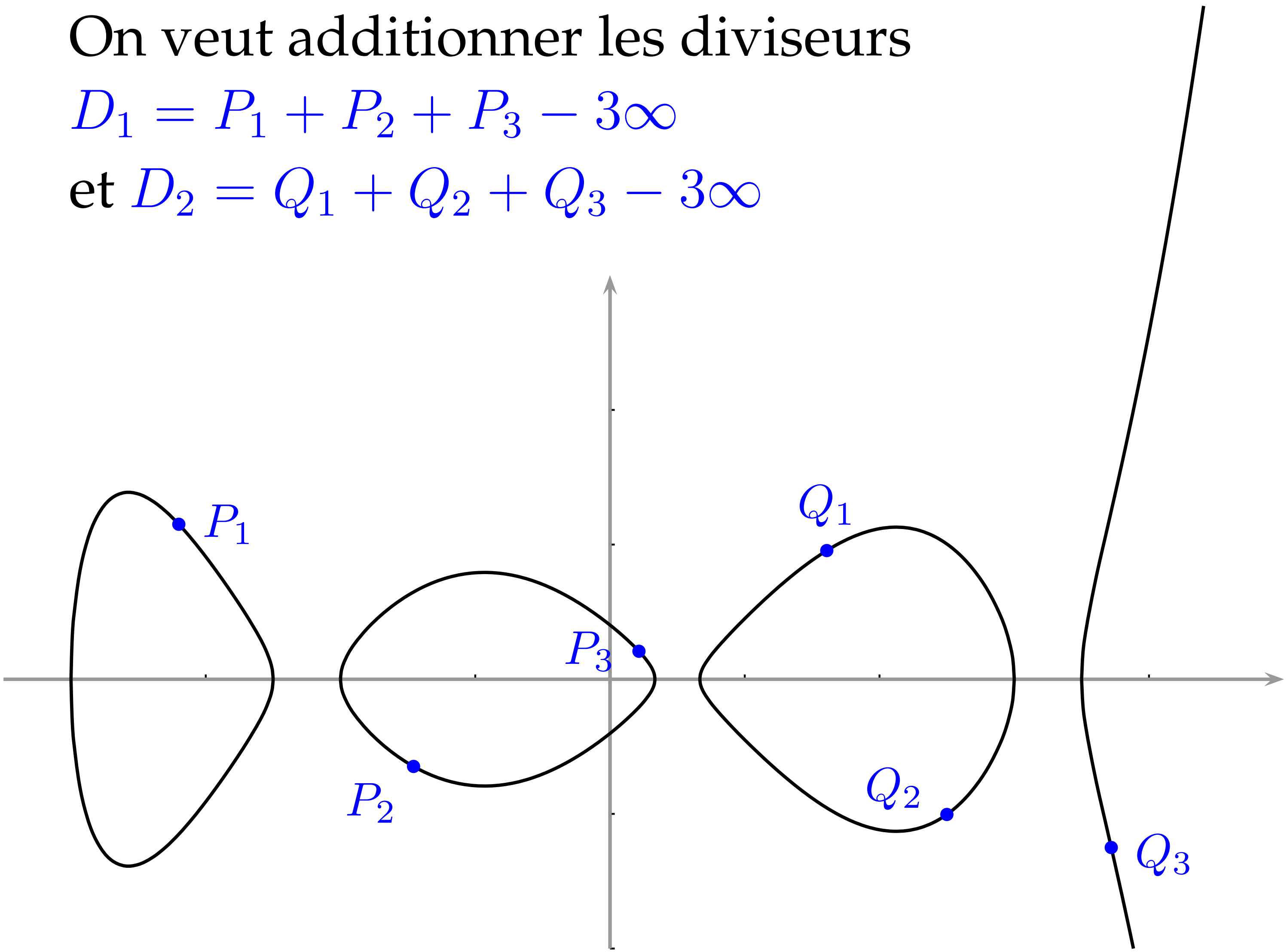


Courbe de genre 3

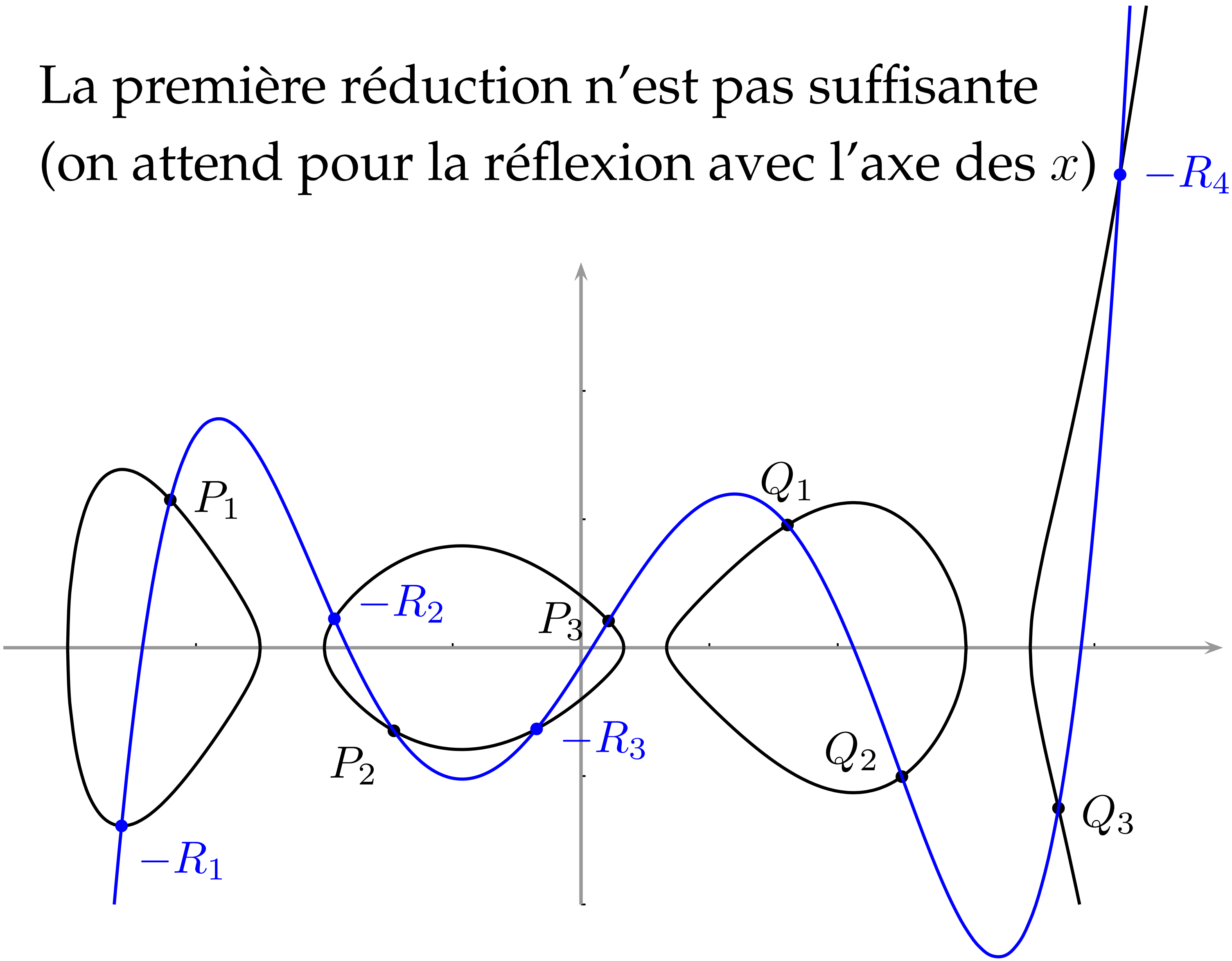
$$y^2 = x^7 + \frac{1}{2}x^6 - \frac{847}{144}x^5 - \frac{325}{144}x^4 + \frac{1763}{192}x^3 + \frac{403}{144}x^2 - \frac{1667}{576}x + \frac{35}{96}$$



On veut additionner les diviseurs
$D_1 = P_1 + P_2 + P_3 - 3\infty$
et $D_2 = Q_1 + Q_2 + Q_3 - 3\infty$



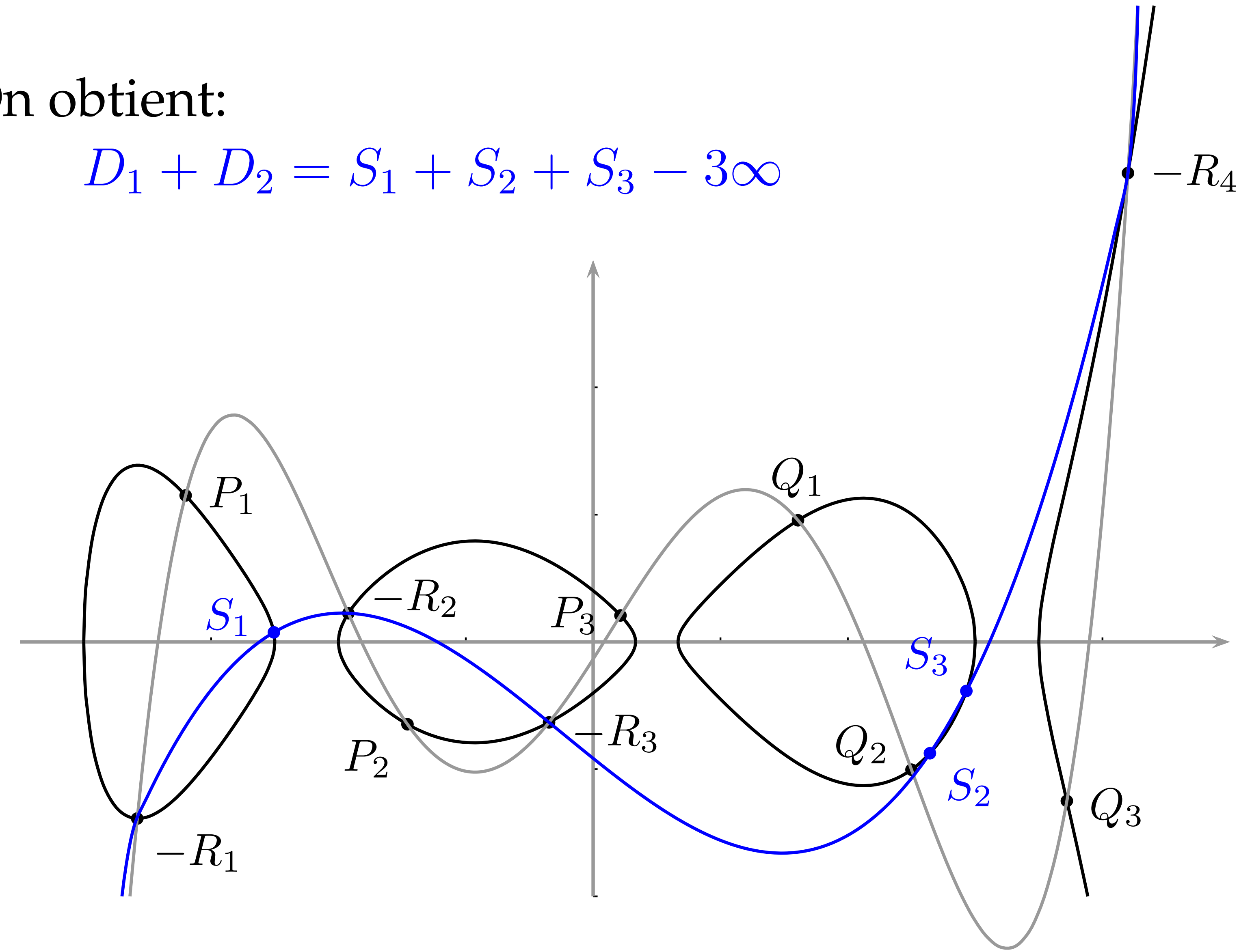
La première réduction n'est pas suffisante
(on attend pour la réflexion avec l'axe des x)



Courbe de genre 3

On obtient:

$$D_1 + D_2 = S_1 + S_2 + S_3 - 3\infty$$



Ring of Polynomials

We consider at the ring of polynomials

$$R = \frac{\mathbb{F}_q[x, y]}{(y^2 + h(x)y - f(x))}$$

and we look at ideals of this ring.

The ideal

$$I = (p_1(x, y), p_2(x, y))$$

is the set of all polynomials of the form

$$r_1(x, y)p_1(x, y) + r_2(x, y)p_2(x, y) \mod y^2 + h(x)y - f(x).$$

p_1 and p_2 are the generators of I.

Ideals

The ideals of R form an infinite multiplicative group.

A principal ideal is an ideal with a single generator, for example $(y - 3x^2 + 8x - 4)$.

The principal ideals of R are a normal subgroup of the ideals of R.

The ideal class group is the group:

$$\frac{\text{ideals of } R}{\text{principal ideals of } R}$$

This is a finite multiplicative group.

Ideal Classes

Each class of ideals contains a unique reduced ideal of the form

$$I = (u(x), y - v(x))$$

with $\deg(u) \leq g$, u monic and $\deg(v) < \deg(u)$.
(By construction, $u(x)$ divides $v(x)^2 + h(x)v(x) - f(x)$.)

For hyperelliptic curves, the ideal class group is isomorphic to the divisor class group ($Jac(C)(\mathbb{F}_q)$).

Working with the ideal class group is easier!!!

Why HEC?

The group order of a curve of genus g over a field of q elements is:

$$|Jac(C)(\mathbb{F}_q)| = q^g + O\left(gq^{g-1/2}\right),$$

so to have the same group order as ECC, we divide the number of bits of the field order by g.

Field multiplications are then $\sim g^2$ times faster (and use less energy).

On the other hand, a group operation takes $O\left(g^2\right)$ field operations.

At a first glance, the difference should be small.

Composition

Input: ideals $I_1 = (u_1(x), y - v_1(x))$ and $I_2 = (u_2(x), y - v_2(x))$.

Output: ideal $I_C = (u_C(x), y - v_C(x))$ (not reduced).

Composition

Input: ideals $I_1 = (u_1(x), y - v_1(x))$ and $I_2 = (u_2(x), y - v_2(x))$.

1. $d_1 = s_1 u_1 + s_2 u_2 \leftarrow gcd(u_1, u_2)$.
2. $d = t_1 d_1 + t_2(v_1 + v_2 + h) \leftarrow gcd(d_1, v_1 + v_2 + h_2)$.
3. $r_1 \leftarrow s_1 t_1$, $r_2 \leftarrow s_2 t_1$, and $r_3 \leftarrow t_2$.

Output: ideal $I_C = (u_C(x), y - v_C(x))$ (not reduced).

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3. $r_1 \leftarrow s_1 t_1$, $r_2 \leftarrow s_2 t_1$, and $r_3 \leftarrow t_2$.
4. $u_C \leftarrow u_1 u_2 / d^2$.
5. $v_C \leftarrow v_2 + \frac{u_2}{d} r_2(v_1 + v_2) + r_3 \frac{v_2^2 + h v_2 + f}{d}$.

Output: ideal $I_C = (u_C(x), y - v_C(x))$ (not reduced).

Reduction

Input: ideal $I_C = (u_C(x), y - v_C(x))$.

Output: reduced ideal $I_3 = (u_3(x), y - v_3(x))$.

Reduction

Input: ideal $I_C = (u_C(x), y - v_C(x))$.

1. $\tilde{u}_0 \leftarrow u_C$, $\tilde{v}_0 \leftarrow v_C$.

Output: reduced ideal $I_3 = (u_3(x), y - v_3(x))$.

Input: ideal $I_C = (u_C(x), y - v_C(x))$.

1. $\tilde{u}_0 \leftarrow u_C$, $\tilde{v}_0 \leftarrow v_C$.
2. From $i = 0$, while $\deg(\tilde{u}_i) > g$:

Output: reduced ideal $I_3 = (u_3(x), y - v_3(x))$.

Input: ideal $I_C = (u_C(x), y - v_C(x))$.

1. $\tilde{u}_0 \leftarrow u_C$, $\tilde{v}_0 \leftarrow v_C$.
2. From $i = 0$, while $\deg(\tilde{u}_i) > g$:
   (a) $\tilde{u}_{i+1} \leftarrow Monic\left(\frac{\tilde{v}_i^2 + h\tilde{v}_i + f}{\tilde{u}_i}\right)$.
   (b) $\tilde{v}_{i+1} \leftarrow \tilde{v}_i + h \bmod \tilde{u}_{i+1}$.
   (c) $i \leftarrow i + 1$.

Output: reduced ideal $I_3 = (u_3(x), y - v_3(x))$.

Reduction

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   (c) $i \leftarrow i + 1$.
3. $u_3 \leftarrow \tilde{u}_i$, $v_3 \leftarrow \tilde{v}_i$.

Output: reduced ideal $I_3 = (u_3(x), y - v_3(x))$.

Attacks for HEC

- Weil descent attack:
  - Frey (1998), Gaudry–Hess–Smart (2000), ...
  - Gaudry (2004)
- Index calculus attack for large genus:
  - Adleman–DeMarrais–Huang (1999)
  - Enge-Gaudry-Thomé (2009)
- Index calculus attack for small genus:
  - Gaudry (2000), Harley (2000), T. (2003)
  - Gaudry–Thomé–T.–Diem (2007).
  - Diem (2006): non-hyperelliptic curves

- Use isomorphisms to choose a form of the curve equation that reduces the cost of the group operation
- Assume the fastest known attack
- The secret key size (scalar) depends only on the security level, not the group order

genus	1	2	3	4
fields size (bits)	n	$n/2$	$3n/8$	$n/3$
group size (bits)	n	n	$9n/8$	$4n/3$
key size (bits)	n	n	n	n

Improving the formulæ

1. Work based on the coefficients instead of polynomials (explicit formulæ)

2. Combine inversions

3. Reduce the number of multiplications

   (a) Faster algorithms

   (b) Karatsuba-like tricks